

VIBRATION OF A CONTINUUM EXCITED BY RANDOM MOTIONS OF A CONTINUOUS FOUNDATION

Stephen H. Crandall

Paul J. Remington

May 15, 1969

Report No. 76205-2

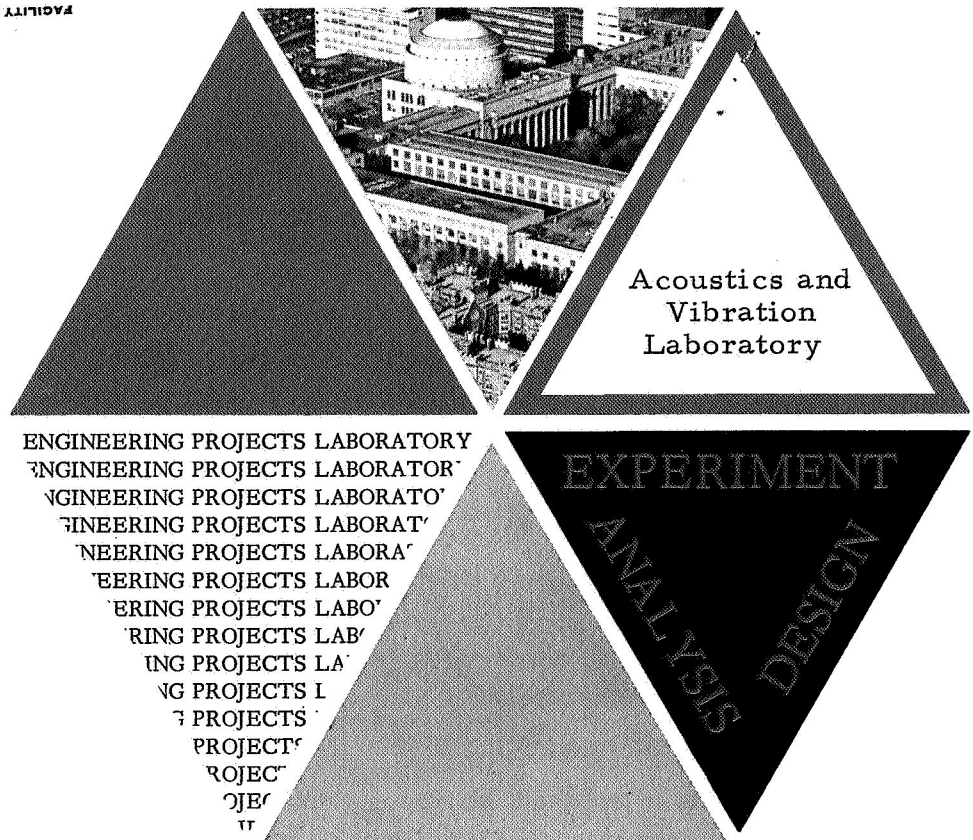
Contract No. NGR22-009-135

Acoustics and Vibration Laboratory
Massachusetts Institute of Technology
Cambridge, Massachusetts

This research was supported by the
National Aeronautics and Space under
Grant NGR22-009-135, and by the
National Science Foundation.

N69-80475
 (ACCESSION NUMBER)
 18
 (PAGES)
 CP-106798
 (NASA CR OR TMX OR AD NUMBER)
 (THRU)
 NONE
 (CODE)
 (CATEGORY)

FACILITY FORM 602



MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Mechanical Engineering

August 12, 1969

To: NASA Office of Grants and Research Contracts
Washington 25, D.C.

Mr. David C. Driscoll
M.I.T.
E19-702

Mr. Harry L. Runyan
NASA
Langley Field
Newport News, Virginia

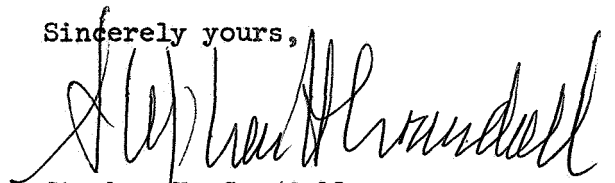
Mr. Phillip Edge
NASA
Langley Field
Newport News, Virginia

Mr. J. M. Cawthorn
NASA
Langley Field
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Gentlemen:

Enclosed please find the Seventh Semiannual Report for NASA Research Grant NGR 22-009-135. This is for the six month period ending June 30, 1969.

Sincerely yours,



Stephen H. Crandall
Professor of Mechanical Engineering

Enclosures

Seventh Semiannual Status Report

June 30, 1969

NASA Research Grant NGR 22-009-135

M.I.T.

Response of Building Structures to Environmental Noise
of Seismic, Acoustic and Aerodynamic Origin

During the past six months we have accomplished a good deal under this project. Two technical papers have been completed and a third is well on the way. One doctor's thesis has reached the rough draft stage. Our experimental facility is now complete and a fairly steady flow of experimental results is now pouring out.

The report DSR 76205-2 entitled "Vibration of a Continuum Excited by Random Motions of a Continuous Foundation" by S. H. Crandall and P. J. Remington (copy attached) was issued on May 15, 1969. The material in this report was presented by Professor Crandall at the Romanian National Conference on Applied Mechanics in Bucharest, June 23-27 and was submitted for publication in the Romanian Journal of Applied Mechanics. No travel expenses were charged to the project.

The paper "On the Use of Slowness Diagrams to Represent Wave Reflections" by S. H. Crandall (abstract attached) was completed and is to be submitted for publication in the Journal of the Acoustical Society of America. This paper on the nature of waves in a half-space is a by-product of our work on waves in structures coupled to half-spaces.

Mr. Remington's thesis "Response of a Plate to Noise in a Supporting Viscoelastic Medium" has reached the rough draft stage and it is expected that he will complete his degree requirements in the summer term. He plans to present his results at the 78th Meeting of the Acoustical Society of America to be held in San Diego, November 4-7, 1969. The Abstract of his presentation entitled "Response of a Plate to Noise in a Supporting Elastic Medium" (see copy attached) will subsequently appear in the Journal of the Acoustical Society.

Mr. Kurzweil has successfully measured the entire matrix of impedance coefficients of the soil surface of our model soil facility. This matrix relates horizontal and vertical force components and rocking moment components applied to a hypothetical rigid massless disk in contact with the surface to the translational and rocking velocities of the disk. The impedance coefficients were derived from measurements on three disks (with mass) of differing sizes. In the frequency range observed (100 - 1000 Hz.) there was small, but measurable, coupling between horizontal translation and rocking.

Mr. Kurzweil is now using this data to predict the response of a column-footing combination. These predictions will then be compared with direct measurements.

Mr. Nigam, after making a survey of pile-foundation structures, decided to concentrate his efforts on the problem of noise transfer between the soil and partially embedded foundation structures representative of actual floating foundations. His thesis proposal (copy attached) was accepted May 29, 1969.

Professor Crandall is completing the task of editing the report "Dynamic Properties of Modelling Clay" which describes our measurements of the basic dynamic properties of Plasticine.

TECHNICAL REPORT NO. 76205-2

VIBRATION OF A CONTINUUM EXCITED BY
RANDOM MOTIONS OF A CONTINUOUS FOUNDATION

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Stephen H. Crandall

and

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This research was supported by the National Aeronautics and Space Administration under Grant NGR22-009-135, and by the National Science Foundation.

May 15, 1969

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Stephen H. Crandall
and Paul J. Remington

Department of Mechanical Engineering
Massachusetts Institute of Technology

ABSTRACT

Prior to placing a dynamic continuum in contact with a vibrating foundation the displacements of the interfacial plane of the foundation constitute a random process in space and time whose characteristics are given. It is desired to predict the characteristics of the interfacial displacement process which occurs after the continuum is placed in contact with the foundation. This represents an idealization of the problem of predicting the ground-excited motions of a structure from measurements of ground motion prior to the erection of the structure. A generalized analysis of the problem is presented for the case where the underlying excitation is stationary in time. This is then specialized to the case where the motions at the interface are spatially homogeneous. The general nature of the problem is exhibited in a simplified manner by studying the case of a membrane stretched over a vibrating viscoelastic Winkler foundation. Extensions to plates on elastic and viscoelastic half-spaces are discussed.

1. Introduction

This paper treats an idealization of the general problem of predicting random vibration levels in a structure from measurements of the existing noise field at the proposed site of the structure. Because of the dynamic interaction between the structure and the foundation the noise field at the interface can be altered considerably by the presence of the structure. The discussion here is limited to linear continuous structures and linear continuous foundations with plane interfacial surfaces. We focus our attention on the random motions at the surface of the interfacial plane before and after the continuous structure is placed in contact with the foundation. In the sequel the continuous structure is called simply the continuum.

2. Description of the interface motion

Let the foundation occupy the half-space $z \leq 0$ and let the interface between the foundation and the proposed continuum be a simply connected region A in the xy plane. Let the position vector from the origin to the point $(x, y, 0)$ be denoted by \vec{r} . In general the displacement of the surface at this point at time t will have three components u , v and w . For simplicity of exposition we confine our attention to those cases where tangential interfacial slip is permitted and only the normal displacement $w(\vec{r}, t)$ is effective in the dynamic interaction. The formal extension to cases with bounded interfaces where $u(\vec{r}, t)$ and $v(\vec{r}, t)$ must be included is not difficult.

We take $w(\vec{r}, t)$, the normal displacement of the interface in the direction of positive z , to be a random process in space and time. The mechanisms responsible for this motion may be microseismic disturbances,

road and rail vehicle noise, machinery vibrations or other urban noises. We assume that the excitation can be taken to be statistically stationary and that measurements of $w(\vec{r}, t)$ can be made prior to the erection of the proposed continuum. We suppose that the ensemble average of w , at all times and at all locations, is zero; i.e., $E[w(\vec{r}, t)] = 0$. We then focus our attention on the space-time correlation of w which provides sufficient statistical information for almost all practical purposes independently of the nature of the process. If it can also be assumed that the process is Gaussian then the space-time correlation provides a complete description of the process. The space-time correlation function

$$R(\vec{r}_1, \vec{r}_2, \tau) = E[w(\vec{r}_1, t) w(\vec{r}_2, t + \tau)] \quad (1)$$

is a function of \vec{r}_1 , \vec{r}_2 and τ but, because of the stationarity, it is independent of t . It is often more convenient to deal with the cross-spectral density $C(\vec{r}_1, \vec{r}_2, \omega)$ which is a Fourier transform of (1)

$$C(\vec{r}_1, \vec{r}_2, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\vec{r}_1, \vec{r}_2, \tau) e^{-i\omega\tau} d\tau \quad (2)$$

In general the interface motion is not spatially homogeneous. If the contact region A extends to infinity in all directions it may happen that (1) and (2) do not depend on \vec{r}_1 and \vec{r}_2 independently but only on their relative displacement $\vec{r}_2 - \vec{r}_1 = \vec{\lambda}$. In this case the random process $w(\vec{r}, t)$ is said to be homogeneous in space and, in place of (1) and (2), we write

$$\begin{aligned} R(\vec{\lambda}, \tau) &= E[w(\vec{r}, t) w(\vec{r} + \vec{\lambda}, t + \tau)] \\ C(\vec{\lambda}, \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\vec{\lambda}, \tau) e^{-i\omega\tau} d\tau \end{aligned} \quad (3)$$

Here a further alternative description is obtained by performing a Fourier transformation with respect to the displacement $\vec{\lambda}$. Introducing the vector wavenumber variable \vec{k} , we define the wavenumber-frequency spectrum $\phi(\vec{k}, \omega)$ as follows.

$$\begin{aligned}\phi(\vec{k}, \omega) &= \frac{1}{4\pi^2} \int_{A \rightarrow \infty} c(\vec{\lambda}, \omega) e^{i\vec{k} \cdot \vec{\lambda}} dA \\ &= \frac{1}{8\pi^3} \int_{-\infty}^{\infty} d\tau \int_{A \rightarrow \infty} R(\vec{\lambda}, \tau) e^{-i(\omega\tau - \vec{k} \cdot \vec{\lambda})} d\vec{\lambda}\end{aligned}\quad (4)$$

where the integration extends over the entire $\vec{\lambda}$ -plane. If it happens that the wavenumber dependence of $\phi(\vec{k}, \omega)$ is a function only of the magnitude k of the wavenumber (i.e., is independent of the vector direction) then the random process $w(\vec{r}, t)$ is said to be isotropic as well as homogeneous and its second order statistical properties are defined by $\phi(k, \omega)$.

The integral of the wavenumber-frequency spectrum $\phi(\vec{k}, \omega)$ over the entire wavenumber plane and over all frequencies gives the mean square of the process $w(\vec{r}, t)$. It is often convenient to work with the function of frequency which remains after integration over wavenumber. We call

$$S(\omega) = \int_{B \rightarrow \infty} \phi(\vec{k}, \omega) dB \quad (5)$$

the frequency spectrum of the stationary homogeneous random process $w(\vec{r}, t)$. In Eq. (5) dB represents the element of area in the wavenumber plane and the integral extends over the entire wavenumber plane.

3. Interaction of foundation and continuum

Let $w_0(\vec{r}, t)$ denote the interface displacement of the foundation prior to the installation of the continuum and let the corresponding statistical descriptions be designated by R_0 , C_0 , ϕ_0 and S_0 . After the continuum has been installed an interaction pressure $p(\vec{r}, t)$ will develop between the foundation and the continuum in order to maintain the same displacement $w(\vec{r}, t)$ in the foundation and in the continuum at all points of the contact region A. The problem we are concerned with is essentially that of predicting the statistical properties of $w(\vec{r}, t)$ given the corresponding properties of $w_0(\vec{r}, t)$. We consider the interaction pressure $p(\vec{r}, t)$ to be entirely dynamic arising only as a reaction to the existing seismic excitation. Because of the assumed linearity the reaction to any nondynamic loading such as gravity could be superposed on our solution.

In order to describe the dynamic properties of the foundation in a formal way we introduce the Green's function $g_f(\vec{r}, t; \vec{r}', t')$ which denotes the foundation displacement at the position \vec{r} and time t due to a unit impulse of force in the ~~position~~ ^{POSITIVE} z-direction applied (to the quiescent foundation) at the position \vec{r}' and time t' . Then by superposition the displacement of the foundation after the continuum is in place is

$$w(\vec{r}, t) = w_0(\vec{r}, t) - \int_{-\infty}^t dt' \int_A g_f(\vec{r}, t; \vec{r}', t') p(\vec{r}', t') dA' \quad (6)$$

Similarly, if $g_c(\vec{r}, t; \vec{r}', t')$ denotes the displacement of the continuum alone at the position \vec{r} and time t due to a unit impulse of force in the positive z-direction applied at the position \vec{r}' and time t' , the displacement of the continuum when it is in place on top of the vibrating foundation is

$$w(\vec{r}, t) = \int_{-\infty}^t dt' \int_A g_c(\vec{r}, t; \vec{r}', t') p(\vec{r}', t') dA' \quad (7)$$

Equations (6) and (7) are simultaneous linear integral equations for the resulting interface displacement $w(\vec{r}, t)$ and pressure $p(\vec{r}, t)$. In particular cases, as we shall see, it is possible to solve these equations to obtain explicit expressions for $w(\vec{r}, t)$ in terms of linear operations on $w_0(\vec{r}, t)$. More generally we can say that Eqs. (6) and (7) are equivalent to the statement that the response displacement $w(\vec{r}, t)$ is related to the entire past history of the excitation w_0 over the complete interface A by some linear functional; e.g.,

$$w(\vec{r}, t) = \mathcal{F}[w_0(\vec{r}', t')] \quad (8)$$

Then proceeding formally

$$w(\vec{r}_1, t) w(\vec{r}_2, t + \tau) = \mathcal{L}_1[w_0(\vec{r}_1', t_1')] \mathcal{L}_2[w_0(\vec{r}_2', t_2' + \tau)] \quad (9)$$

$$= \mathcal{L}_1[w_0(\vec{r}_1', t_1')] \mathcal{L}_2[w_0(\vec{r}_2', t_2' + \tau)]$$

and by interchanging the order of ensemble averaging with evaluating the linear functional we have

$$R(\vec{r}_1, \vec{r}_2, \tau) = \mathcal{L}_1 \mathcal{L}_2 [R_0(\vec{r}_1', \vec{r}_2', \tau - t_1' + t_2')] \quad (10)$$

as a formal indication of how the space-time correlation of the response $w(\vec{r}, t)$ depends on the space-time correlation of the excitation $w_0(\vec{r}, t)$.

A more explicit representation can be obtained in case w_0 and w are both spatially homogeneous. This requires that both the foundation and the continuum have uniform dynamic properties (independent of the position \vec{r}) and that the interfacial region A be unbounded. We introduce the space-time Fourier transform for the displacement $w(\vec{r}, t)$

$$W(\vec{k}, \omega) = \int_{-\infty}^{\infty} dt \int_{A \rightarrow \infty} w(\vec{r}, t) e^{-i(\omega t - \vec{k} \cdot \vec{r})} dA \quad (11)$$

where the space integral is over the entire plane. The Fourier transforms $W_0(\vec{k}, \omega)$ and $P(\vec{k}, \omega)$ for $w_0(\vec{r}, t)$ and $p(\vec{r}, t)$ are defined analogously. When the dynamic properties of the foundation are invariant in space and time the Green's function $g_f(\vec{r}, t; \vec{r} - \vec{\lambda}, t - \tau)$ is independent of \vec{r} and t and we can write

$$g_f(\vec{r}, t; \vec{r} - \vec{\lambda}, t - \tau) = h_f(\vec{\lambda}, \tau) \quad (12)$$

for the impulse response of the foundation with a corresponding definition for $h_c(\vec{\lambda}, \tau)$, the impulse response of the continuum. The wavenumber-frequency response of the foundation may be defined as the Fourier transform of the impulse response

$$H_f(\vec{k}, \omega) = \int_{-\infty}^{\infty} d\tau \int_{A \rightarrow \infty} h_f(\vec{\lambda}, \tau) e^{-i(\omega \tau - \vec{k} \cdot \vec{\lambda})} dA \quad (13)$$

This can also be interpreted as the traveling wave response of the foundation. Similarly $H_c(\vec{k}, \omega)$ is defined as the wavenumber-frequency response or traveling wave response of the continuum.

With these definitions the Fourier transforms of Eqs. (6) and (7) are

$$W(\vec{k}, \omega) = W_o(\vec{k}, \omega) - H_f(\vec{k}, \omega) P(\vec{k}, \omega) \quad (14)$$

$$W(\vec{k}, \omega) = H_c(\vec{k}, \omega) P(\vec{k}, \omega)$$

which can be solved simultaneously to yield

$$W(\vec{k}, \omega) = H(\vec{k}, \omega) W_o(\vec{k}, \omega) \quad (15)$$

where

$$H(\vec{k}, \omega) = \frac{H_c(\vec{k}, \omega)}{H_c(\vec{k}, \omega) + H_f(\vec{k}, \omega)} \quad (16)$$

is the traveling wave response at the interface of the composite system due to the same excitation which would cause a unit traveling wave on the foundation alone. The corresponding impulse response is the inverse Fourier transform

$$h(\vec{\lambda}, \tau) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} d\omega \int_{B \rightarrow \infty} H(\vec{k}, \omega) e^{i(\omega\tau - \vec{k} \cdot \vec{\lambda})} dB \quad (17)$$

where dB represents the element of area in the wavenumber plane and the integration extends over the entire plane. The inverse Fourier transform of (15) is

$$w(\vec{r}, t) = \int_0^{\infty} d\tau \int_{A \rightarrow \infty} h(\vec{\lambda}, \tau) w_o(\vec{r} - \vec{\lambda}, t - \tau) dA \quad (18)$$

which is a particular example of the general representation (8). The general relation (10) between input and output space-time correlation functions becomes, in the case of spatial homogeneity,

$$R(\vec{\lambda}, \tau) = \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_{A_1 \rightarrow \infty} dA_1 \int_{A_2 \rightarrow \infty} dA_2 h(\vec{\lambda}_1, \tau_1) h(\vec{\lambda}_2, \tau_2) R_0(\vec{\lambda} - \vec{\lambda}_1 + \vec{\lambda}_2, \tau - \tau_1 + \tau_2) \quad (19)$$

The corresponding relation for the wavenumber-frequency spectra defined by (4) is obtained by Fourier transformation of (19) and (17)

$$\phi(\vec{k}, \omega) = |H(\vec{k}, \omega)|^2 \phi_0(\vec{k}, \omega) \quad (20)$$

Finally if both the excitation spectrum and the composite system are isotropic (i.e., if the wavenumber dependence of both ϕ_0 and H are functions of k rather than \vec{k}) then the response spectrum is also isotropic.

4. Membrane on viscoelastic Winkler foundation

As a simple example to illustrate the application of the preceding analysis we consider the following case. The foundation is taken to be the infinite halfspace $z \leq 0$ with an ideal Winkler-type constitutive relation. When a tensile load $-p(\vec{r}, t) dA$ is applied to the free surface $z = 0$ the resulting surface displacement $w(\vec{r}, t)$ within the loaded area dA satisfies a Voight-type constitutive relation

$$-p = b \frac{\partial w}{\partial t} + cw \quad (21)$$

where b is a damping constant and c is a spring constant. It is assumed that the surface remains undeformed outside the loaded area dA .

We consider the case where initially the surface of the foundation has a random displacement $w_0(\vec{r}, t)$ which is stationary in time and both homogeneous and isotropic in space. The wavenumber-frequency spectrum $\phi_0(k, \omega)$ for the initial displacement process is assumed to be known. We now pose the problem of predicting the wavenumber-frequency spectrum $\phi(k, \omega)$ of the interface process $w(\vec{r}, t)$ after a membrane has been stretched over the foundation. The membrane which plays the role of the continuum in the preceding section is taken to have surface tension T per unit length and mass m per unit area so that the equation of motion for the membrane alone is

$$-T \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + m \frac{\partial^2 w}{\partial t^2} = p \quad (22)$$

where $p(\vec{r}, t)$ is the pressure loading acting on the bottom surface of the membrane. When the membrane is stretched over the foundation we assume that bonding occurs so that both the membrane and the foundation are constrained to share the same displacement $w(\vec{r}, t)$. As a result an interference loading $p(\vec{r}, t)$ is developed and the original displacement process $w_0(\vec{r}, t)$ is modified.

To apply the analysis given previously we first calculate the traveling wave response of the foundation and of the membrane. Assuming a unit traveling wave excitation of the form $\exp\{i(\omega t - \vec{k} \cdot \vec{r})\}$ we obtain the foundation wavenumber-frequency response function (13) directly from (21)

$$H_f(\vec{k}, \omega) = \frac{1}{i\omega b + c} \quad (23)$$

In the same way the membrane traveling wave response is obtained from (22)

$$H_c(\vec{k}, \omega) = \frac{1}{Tk^2 - m\omega^2} \quad (24)$$

Note that in this case both the foundation and the continuum are isotropic.

The traveling wave response of the composite system, according to (15) is

$$\begin{aligned} H(k, \omega) &= \frac{H_c}{H_c + H_f} \\ &= \frac{i\omega b + c}{Tk^2 + c + i\omega b - m\omega^2} \end{aligned} \quad (25)$$

The resulting interface wavenumber-frequency spectrum is then given by (20). To carry this example somewhat farther we assume that the original wavenumber-frequency spectrum $\phi_o(k, \omega)$ has the following special form

$$\phi_o(k, \omega) = \begin{cases} \frac{1}{\pi k_o^2} S_o(\omega), & 0 < k < k_o \\ 0, & k_o < k \end{cases} \quad (26)$$

i.e., we assume that the original wavenumber-frequency spectrum is flat in wavenumber up to a cut-off wavenumber k_o . Note that according to (5) the frequency spectrum of the original process is $S_o(\omega)$. We shall determine the corresponding frequency spectrum $S(\omega)$ of the resulting interface process in the composite system.

Inserting (25) and (26) into (20) we obtain

$$\phi(k, \omega) = \frac{S_0(\omega)}{\pi k_0^2} \frac{1 + 4\zeta^2 \omega^2/\omega_0^2}{(1 - \omega^2/\omega_0^2 + \epsilon k^2/k_0^2)^2 + 4\zeta^2 \omega^2/\omega_0^2} \quad (27)$$

where

$$\omega_0^2 = \frac{c}{m}, \quad 2\zeta = \frac{b}{\sqrt{mc}}, \quad \epsilon = \frac{k_0^2 T}{c} \quad (28)$$

The parameters ω_0 and ζ are the undamped natural frequency and damping ratio for the mode of uniform vibration in which the membrane bounces on the foundation without deforming. The dimensionless parameter ϵ measures the cut-off wavenumber of the excitation in terms of the membrane surface tension and the foundation stiffness. It can be interpreted in a simple manner by imagining a sinusoidal static load with wavenumber k_0 applied separately to the foundation and to the membrane. In both cases the static deformation is also sinusoidal with wavenumber k_0 . If the load amplitudes are adjusted so as to make the displacement amplitudes equal then the parameter ϵ is just the ratio of the membrane load amplitude to the foundation load amplitude.

The frequency response of the composite system is obtained by inserting (27) into (5)

$$\begin{aligned} S(\omega) &= \int_0^{k_0} \phi(k, \omega) 2\pi k dk \\ &= S_0(\omega) \frac{1}{\epsilon} \frac{1 + 4\zeta^2 \omega^2/\omega_0^2}{2\zeta\omega/\omega_0} (\theta_2 - \theta_1) \end{aligned} \quad (29)$$

where the angles θ_1 and θ_2 are given by

$$\begin{aligned}\tan \theta_1 &= \frac{1 - \omega^2/\omega_o^2}{2\zeta\omega/\omega_o} \\ \tan \theta_2 &= \frac{1 - \omega^2/\omega_o^2 + \epsilon}{2\zeta\omega/\omega_o}\end{aligned}\tag{30}$$

It follows from (29) that S/S_o approaches $(1 + \epsilon)^{-1}$ when $\omega \rightarrow 0$ and that S/S_o approaches zero when $\omega \rightarrow \infty$. The nature of the variation of S/S_o with frequency is indicated in Fig. 1 for $\epsilon = 1$ and $\epsilon = 5$ when $\zeta = 0.1$. The effect of decreasing the damping is to raise the levels in the pass-bands (in inverse proportion to ζ) without significantly changing the bandwidths.

5. Other applications

The analytical framework developed in Sec. 3 can be applied to a wide range of foundation and continuum models. For example it is a simple matter to restrict the models to one spatial dimension and treat beams and strings on viscoelastic foundation. The two-dimensional model studied in Sec. 4 can be made more representative of structural practice by replacing the Winkler foundation by a continuous half-space with distributed elastic and viscous parameters and by replacing the membrane by a plate. In the case of a Bernoulli-Euler plate on a homogeneous isotropic elastic half-space [1] the traveling wave response of the foundation (13) is

$$H_f(k, \omega) = \frac{i \sqrt{\omega^2/c_1^2 - k^2} \omega^2/c_2^2}{G[(2k^2 - \omega^2/c_2^2)^2 + 4k^2 \sqrt{\omega^2/c_1^2 - k^2} \sqrt{\omega^2/c_2^2 - k_2^2}]} \tag{31}$$

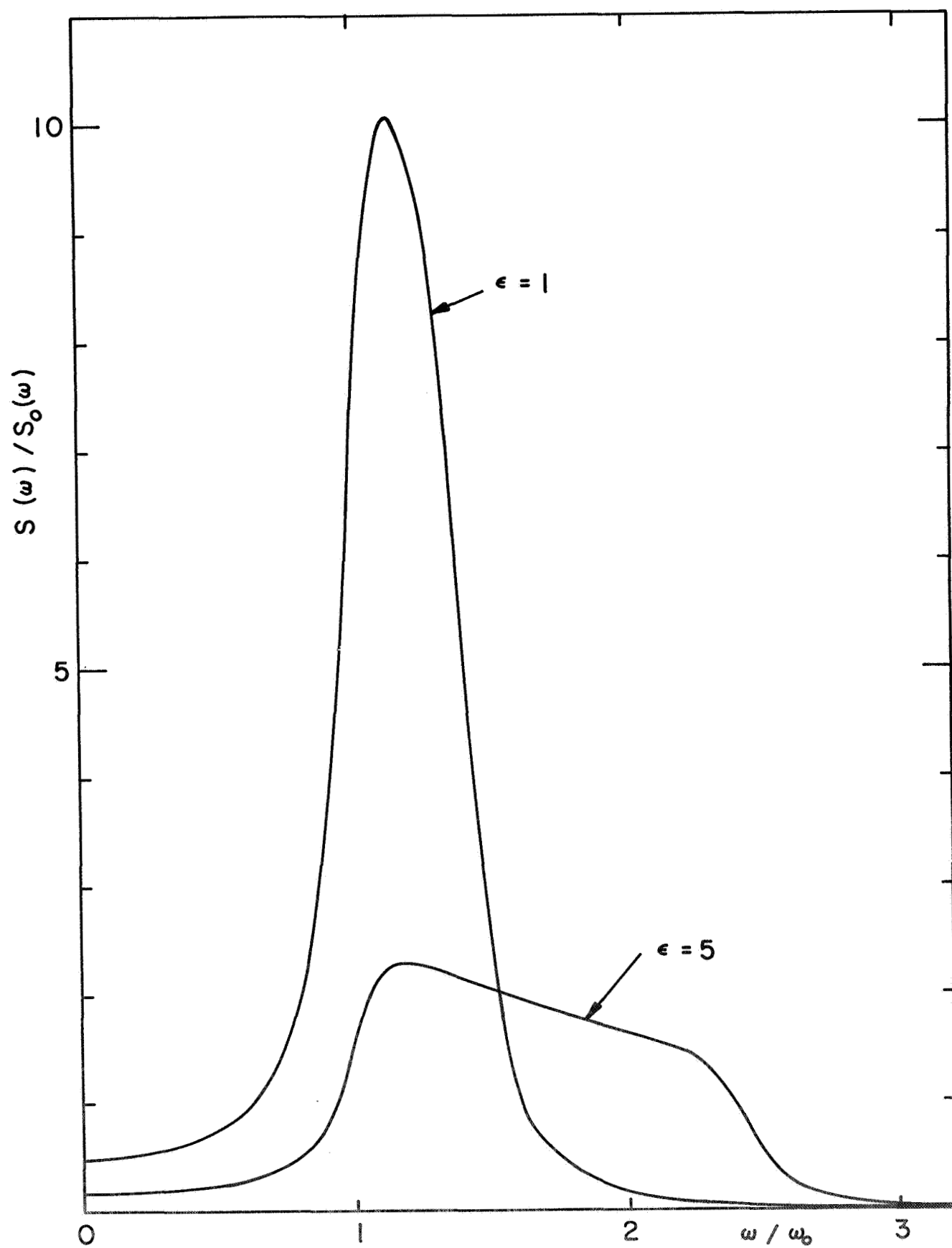


Figure 1 Frequency response of membrane on Winkler foundation ($\zeta = 0.10$) for random excitation with $\phi(k, \omega) = S_0(\omega) / \pi k_0^2$ for $k \leq k_0$ and $\phi(k, \omega) = 0$ for $k_0 < k$.